

Coset symmetries in dimensionally reduced heterotic supergravity

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ABSTRACT: We investigate the coset structures which appear in the dimensional reduction of supergravity theories. Especially we investigate how to recognize the global symmetry groups if the coset is non-split. As an example we apply our analysis to the theories emanating from the dimensional reduction of Heterotic supergravity.

KEYWORDS: Supergravity Models, Global Symmetries.

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Contents

1. Introduction	1
2. The dimensional reduction method	2
2.1 Reduction over one circle	3
2.2 Reduction over d circles	3
3. The algebra from the reduction	5
3.1 Restricted roots and coset Lagrangians	5
3.2 Identifying restricted roots in the Lagrangian	8
4. Analysis of dimensionally reduced symmetries	11
5. Maximal scalar manifolds and dualizations	12
6. Conclusions	13
A. Real forms of Lie algebras: a quick reference	14
B. Some details of the Lie algebra $\mathfrak{so}(d, d + N)$.	15
C. Baker-Campbell-Hausdorff and tricks	16
D. Tables	17

1. Introduction

Supergravity theories provide a useful playground for probing string theory physics (see e.g. [1] and references therein), since they are on the one hand a limit of string theory, but on the other hand they are classical field theories, admitting a more simpler analysis than string theory.

The scenario of dimensional reduction provides us with a setting in which an effective four dimensional theory can be obtained from the ten dimensional supergravity theories, therefore bringing four dimensional physics in contact with string theory. The scalars in supergravity theories often parameterize cosets G/K where K is the maximal compact subgroup of the global symmetry group G . This paper is about the cosets G/K . The global symmetry group G is related to the U-duality group, which contains the S- and T-duality of string theories [2–4].

In maximal supergravities it was shown [5, 6] that in a circle by circle reduction all scalars appeared in upper-triangular matrices parameterizing the solvable positive root

subalgebra of the global symmetry group. Recognizing the group G is greatly facilitated by the fact that the Lie algebra in these reductions of maximal supergravities is a split real form, because then all dilaton coupling vectors can be identified with positive roots and a Dynkin diagram can be drawn.

For non-maximal supergravities the Lie algebra of the global symmetry group can be a non-split real form. In this paper we will address the question of how to recognize the global symmetry group parameterized by the scalars in the case where the Lie algebra is not a split real form. We find that the identification of roots is replaced by the identification of restricted roots and that the Dynkin diagram is that of the restricted root system. Together with the multiplicities of the restricted roots this fixes uniquely the global symmetry group. Theories where scalars parameterize cosets G/K where G is a non-split real form have been studied before, see e.g. [7, 8], but in this work the groups G and K were known beforehand. We present a technique to find the groups G and K in theories obtained from a dimensional reduction.

In this paper we will focus on the dimensional reduction of Heterotic supergravity as a relevant example where non-split real forms arise. The result is known [9–11] but in this paper we present a method which can be used for any supergravity theory and which gives more insight in how the cosets appear in supergravity theories.

The paper is organized as follows; in section 2 we perform the dimensional reduction of Heterotic supergravity to outline the method of dimensional reduction. In section 3 we show the relation between restricted roots and scalar coset Lagrangians; appendix A is a quick reference for Lie algebraic concepts and explains our notational conventions on Lie algebras. In section 4 we analyze the reduction of the higher dimensional symmetries of Heterotic supergravity and show that they give the same symmetry group as obtained by the method of section 3. In section 5 we discuss the concept of a maximal scalar manifold and show for Heterotic supergravities how field dualizations give symmetry enhancements. In section 6 we draw some conclusions.

2. The dimensional reduction method

The method of dimensional reduction used in this paper is similar to that of [5, 6]. Actually a Kaluza-Klein circle by circle reduction is performed; the total number of circles reduced upon is called d and the dimension of the field theory is $D = 10 - d$. Since the global symmetry group manifests itself already on the bosons in the theory and since the Kaluza-Klein procedure does not break supersymmetry, we will only be concerned with the bosonic field content. The fields are: the metric $g_{\mu\nu}$, the Yang-Mills field A_μ in some representation of either $E_8 \times E_8$ or $SO(32)$, the dilaton Φ_0 and the Kalb-Ramond gauge potentials $B_{\mu\nu}$. As usual we will restrict ourselves to the abelian subalgebra of the Yang-Mills sector and therefore only 16 gauge bosons remain, but we will not restrict ourselves to this number 16 and just assume the existence of N abelian gauge bosons.

The action can be written in Einstein frame as

$$S = \int_{\mathcal{M}_{10}} d^{10}x e \left(R - \frac{1}{2}(\partial\Phi_0)^2 - \frac{1}{12}e^{-\Phi_0}H^2 - \frac{1}{4}e^{-\frac{1}{2}\Phi_0} \sum_{I=1}^{16} F_{\mu\nu}^I F^{I\mu\nu} \right), \quad (2.1)$$

where $e = \det(e_{\mu}^a)$ and $F^I = dA^I$. The field strength H contains the Yang-Mills Chern-Simons term: $H = dB - \sum_{I=1}^N \frac{1}{2} A^I \wedge F^I$.

2.1 Reduction over one circle

To obtain the $(10 - d)$ -dimensional theory, we reduce d times over a circle. In going from $D + 1$ to D dimensions we write

$$ds_{D+1}^2 = e^{2\alpha\varphi} ds_D^2 + e^{-2\alpha(D-2)\varphi} (dz + V_{\mu} dx^{\mu})^2, \tag{2.2}$$

where the number α is given by

$$\alpha = \sqrt{\frac{1}{2(D-1)(D-2)}} \equiv \frac{1}{2}s, \tag{2.3}$$

and where z is the coordinate over the circle. All fields are independent of z . The Kaluza-Klein vector is denoted by V_{μ} and reducing over more than one dimension will result in a set of Kaluza-Klein vectors V_{μ}^i . To obtain the action we use the following rules for the dimensional reduction of the Ricci scalar and an $n + 1$ -form field strength F_{n+1} at every step in going from $D + 1$ to D dimensions:

$$\sqrt{-\hat{g}}\hat{R} = \sqrt{-g} \left(R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{4}e^{-(D-1)s\varphi} F(V)^2 \right) \tag{2.4a}$$

$$\frac{1}{2(n+1)!} \sqrt{-\hat{g}}\hat{F}_{(n+1)}^2 = \frac{1}{2(n+1)!} \sqrt{-g}F_{(n+1)}^2 e^{-ns\varphi} + \sqrt{-g} \frac{1}{2n!} F_{(n)}^2 e^{(D-n-1)s\varphi}. \tag{2.4b}$$

This is enough to determine the action in any dimension D . The fields descending from the ten dimensional metric are: the metric $g_{\mu\nu}$, the dilatons φ_i , the Kaluza-Klein vectors V_{μ}^i and the axions, which actually descend from the Kaluza-Klein vectors, A_{ij} and are thus defined only for $i < j$. The ten dimensional Kalb-Ramond gauge potential gives rise to a two-form $B_{\mu\nu}$, vectors $B_{\mu i}$ and scalars B_{ij} . The ten dimensional Yang-Mills field gives rise to a Yang-Mills vector A_{μ}^I and scalars A_i^I .

2.2 Reduction over d circles

The circle by circle reduction from 10 to D dimensions can be seen as a dimensional reduction over a d -torus with the following metric Ansatz:

$$ds_{10}^2 = e^{\frac{1}{2}\vec{g}\cdot\vec{\varphi}} ds_D^2 + \sum_{i=1}^d e^{2\vec{\gamma}_i\cdot\vec{\varphi}} (h^i)^2, \quad d + D = 10, \tag{2.5}$$

where the toroidal coordinates are denoted z^i and

$$\vec{\varphi} = (\varphi_1, \dots, \varphi_d), \quad \vec{g} = 2(s_1, s_2, \dots, s_d), \tag{2.6a}$$

$$\vec{\gamma}_i = \frac{1}{4}\vec{g} - \frac{1}{2}\vec{f}_i, \quad \vec{f}_i = \underbrace{(0, \dots, 0)}_{i-1}, (9-i)s_i, s_{i+1}, \dots, s_d, \tag{2.6b}$$

$$h^i = dz_i + V_{\mu}^i dx^{\mu} + \sum_{i < j \leq d} A_{ij} dz_j. \tag{2.6c}$$

Equation 2.6c can be inverted to give

$$dz^i = \Gamma_{ij}(h^j - V^j) \quad (2.7)$$

where

$$\Gamma_{ij} = \left(\sum_{p=0}^{\infty} (-A)^p \right)_{ij} = \delta_{ij} - A_{ij} + A_{im}A_{mj} - \dots, \quad (2.8)$$

and it should be noted that due to fact that A_{ij} is only defined for $i < j$ the matrix $(A)_{ij} = A_{ij}$ is upper-triangular and hence the series for Γ_{ij} is finite. Using a hat for generic higher dimensional fields and the coordinate split $\hat{x} = (x^\mu, z^i)$ the Kaluza-Klein Ansatz for the Yang-Mills gauge potential is

$$\hat{A}^I(\hat{x}) = A_\mu^I(x)dx^\mu + A_i^I(x)dz^i = A^I + A_i^I dz^i, \quad (2.9)$$

and hence ¹

$$\hat{F}^I = dA^I + dA_i^I dz^i = F^I + F_i^I h^i \quad (2.10a)$$

$$F_i^I = (dA_m^I)\Gamma_{mi} \quad (2.10b)$$

$$F^I = dA^I - F_i^I V^i. \quad (2.10c)$$

For the Kalb-Ramond field we proceed analogously by expanding the two-form field as

$$\hat{B} = B + B_i dz^i + \frac{1}{2} B_{ij} dz^i dz^j, \quad (2.11)$$

and then calculating $d\hat{B}$ and expressing this in the h^i -basis. By incorporating the Chern-Simons terms in a similar fashion one obtains ²

$$\hat{H} = H + H_i h^i + \frac{1}{2} H_{ij} h^i h^j, \quad (2.12a)$$

$$H = dB - (dB_i)\Gamma_{ij} V^j + \frac{1}{2} (dB_{ij})\Gamma_{im}\Gamma_{jn} V^m V^n - \frac{1}{2} (A^I - A_i^I \Gamma_{ij} V^j) F^I, \quad (2.12b)$$

$$H_n = (dB_i)\Gamma_{in} - (dB_{ij})\Gamma_{im}\Gamma_{jn} V^m - \frac{1}{2} (A^I - A_i^I \Gamma_{ij} V^j) F_n^I - \frac{1}{2} A_n^I F^I, \quad (2.12c)$$

$$H_{mn} = (dB_{ij})\Gamma_{im}\Gamma_{jn} + \frac{1}{2} (A_p^I \Gamma_{pm} F_n^I - A_p^I \Gamma_{pn} F_m^I). \quad (2.12d)$$

In an analogous way one obtains for the Kaluza-Klein field strengths:

$$\hat{F}^i = F^i + F_{ij} h^j, \quad (2.13a)$$

$$F_{ij} = (dA_{im})\Gamma_{mj}, \quad (2.13b)$$

$$F^i = dV^i - (dA_{im})\Gamma_{mn} V^n = dV^i - F_{ij} V^j. \quad (2.13c)$$

The dilatons from the reduction φ_i have the usual field strengths $d\varphi_i$. In writing down the action it is convenient to put all dilatons, both from the reduction and the ten dimensional

¹The wedge symbols \wedge will be omitted where possible.

²Any repeated index, whether in an up-down combination or not, is summed over unless otherwise specified.

dilaton, into a $d+1$ -component vector, and hence all dilaton couplings into $d+1$ -component ‘coupling vectors’. The useful definitions are

$$\begin{aligned}
 \vec{\Phi} &= (\Phi_0, \varphi_1, \dots, \varphi_d) = (\Phi_0, \vec{\varphi}), & \vec{F}_i &= (0, f_i), \\
 \vec{B}_{ij} &= -\vec{F}_i + \vec{F}_j, & \vec{G} &= (1, \vec{g}), \\
 \vec{A}_i &= \vec{F}_i - \vec{G}, & \vec{B}_i &= -\vec{F}_i, \\
 \vec{A}_{ij} &= \vec{F}_i + \vec{F}_j - \vec{G}, & \vec{A} &= -\vec{G}, \\
 C_i &= \vec{F}_i - \frac{1}{2}\vec{G},
 \end{aligned} \tag{2.14}$$

of which some summation relations can be deduced

$$\vec{A}_{ij} + \vec{B}_{ik} = \vec{A}_{ki}, \quad \vec{A}_{ij} + \vec{B}_{jk} = \vec{A}_{ik}, \tag{2.15a}$$

$$\vec{C}_i + \vec{C}_j = \vec{A}_{ij}, \quad \vec{C}_j = \vec{B}_{ij} + \vec{C}_i, \quad i < j, \tag{2.15b}$$

and some inner product relations

$$\vec{G} \cdot \vec{G} = \frac{8}{D-2}, \quad \vec{F}_i \cdot \vec{F}_j = 2\delta_{ij} + \frac{2}{D-2}, \tag{2.16a}$$

$$\vec{F}_i \cdot \vec{G} = \frac{4}{D-2}, \quad \vec{A}_i \cdot \vec{G} = -\frac{4}{D-2}, \tag{2.16b}$$

$$\vec{A}_{ij} \cdot \vec{G} = 0, \quad \vec{B}_{ij} \cdot \vec{B}_{kl} = 2\delta_{ik} - 2\delta_{il} - 2\delta_{jk} + 2\delta_{jl}, \tag{2.16c}$$

$$\vec{A}_{ij} \cdot \vec{B}_{kl} = -2\delta_{ik} + 2\delta_{il} - 2\delta_{jk} + 2\delta_{jl}, \tag{2.16d}$$

$$\vec{C}_i \cdot \vec{C}_j = 2\delta_{ij}. \tag{2.16e}$$

The D -dimensional Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \tag{2.17a}$$

$$e^{-1}\mathcal{L}_1 = R - \frac{1}{2}\partial_\mu \vec{\Phi} \cdot \partial^\mu \vec{\Phi} - \frac{1}{2} \sum_{1 \leq i < j \leq d} (F_{ij})^2 e^{\vec{B}_{ij} \cdot \vec{\Phi}} - \frac{1}{4} \sum_{i=1}^d (F_i)^2 e^{\vec{B}_i \cdot \vec{\Phi}} \tag{2.17b}$$

$$e^{-1}\mathcal{L}_2 = -\frac{1}{12} e^{\vec{A} \cdot \vec{\Phi}} H^2 - \frac{1}{4} \sum_{i=1}^d e^{\vec{A}_i \cdot \vec{\Phi}} (H^i)^2 - \frac{1}{2} \sum_{1 \leq i < j \leq d} e^{\vec{A}_{ij} \cdot \vec{\Phi}} (H_{ij})^2 \tag{2.17c}$$

$$e^{-1}\mathcal{L}_3 = -\frac{1}{4} e^{-\frac{1}{2}\vec{G} \cdot \vec{\Phi}} \sum_{I=1}^N (F^I)^2 - \frac{1}{2} \sum_{i=1}^d \sum_{I=1}^N e^{\vec{C}_i \cdot \vec{\Phi}} (F_i^I)^2. \tag{2.17d}$$

3. The algebra from the reduction

3.1 Restricted roots and coset Lagrangians

In this section we will set out the method for identifying the global symmetry group G of the coset G/K which is parameterized by the scalars emanating from a dimensional reduction. A summary of some Lie group theoretical aspects is in appendix A. Every semi-simple real Lie algebra \mathfrak{g} of a Lie group G can be decomposed in a compact subalgebra \mathfrak{k} , a maximal abelian subalgebra \mathfrak{a} and a nilpotent subalgebra \mathfrak{n} . This decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the

Iwasawa decomposition. The orthogonal component of \mathfrak{g} with respect to the Cartan-Killing form $B(x, y)$ to \mathfrak{k} is denoted \mathfrak{p} and we have

$$[\mathfrak{k} \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k} \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p} \mathfrak{p}] \subset \mathfrak{k}. \quad (3.1)$$

If K is a maximal compact subgroup of the Lie group G with Lie algebra \mathfrak{k} , then we can describe the coset G/K by $\exp(\mathfrak{a} \oplus \mathfrak{n})^3$. The scalar manifolds appearing in supergravities are in general Riemannian globally symmetric spaces and can be described by cosets of the form G/K , where K is the maximal compact subgroup of a semi-simple real Lie group G . These cosets G/K are classified (see e.g. [12]). The classification says which maximal compact subalgebras \mathfrak{k} can be found in a real semi-simple Lie algebra \mathfrak{g} and thus which real forms \mathfrak{g} a complex Lie algebra $\mathfrak{g}' \cong \mathfrak{g}^{\mathbb{C}}$ can have. With respect to the subalgebra \mathfrak{a} the Lie algebra \mathfrak{g} can be decomposed into restricted root spaces \mathfrak{g}_λ :

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda, \quad [Hx_\lambda] = \lambda(H)x_\lambda, \quad \forall H \in \mathfrak{a}, x_\lambda \in \mathfrak{g}_\lambda, \quad (3.2)$$

where Σ is the set of (nonzero) restricted roots. This decomposition is analogous to the root decomposition with respect to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of the complexified algebra $\mathfrak{g}^{\mathbb{C}}$, but since the real numbers do not form a closed field, $\mathfrak{a}^{\mathbb{C}}$ can not be identified with $\mathfrak{h}^{\mathbb{C}}$, but only with a subalgebra of the latter. Hence the name restricted root. The restricted roots are linear real functionals on \mathfrak{a} and form a root system [12], which can in fact be non-reduced, i.e. if $\lambda \in \Sigma$ then the only multiples of λ that can also be in Σ are $\pm\lambda, \pm 2\lambda, \pm \frac{1}{2}\lambda$ (but if $\lambda, 2\lambda \in \Sigma$ then $\frac{1}{2}\lambda \notin \Sigma$). Another major deviation from the ordinary roots is that the dimension of the restricted root spaces can exceed 1: $m_\lambda \equiv \dim \mathfrak{g}_\lambda \geq 1$. As with ordinary roots a set of simple restricted roots can be defined and a Dynkin diagram can be drawn. The number of simple restricted roots is called the rank and it equals the dimension of \mathfrak{a} . The multiplicities m_{λ_i} and $m_{2\lambda_i} = \dim \mathfrak{g}_{2\lambda_i}$ of the simple restricted roots λ_i together with the restricted root diagram uniquely fix the real form \mathfrak{g} of a complex Lie algebra $\mathfrak{g}' \cong \mathfrak{g}^{\mathbb{C}}$ and hence they fix the coset G/K . In appendix D we list for all real non-compact forms the restricted root diagram, the multiplicities of the simple restricted roots and the Satake diagram (see appendix A for some explanation).

As seen in the previous section, a circle by circle dimensional reduction reveals the dilaton coupling vectors and these we will identify with the set of positive restricted roots Σ^+ . The multiplicities are easily determined; they are just the number of times a dilaton coupling with that restricted root occurs. A restricted root Dynkin diagram can readily be drawn and thus the corresponding coset can be read off from the tables in appendix D. It only needs a proof that the scalars in the Lagrangian really make up a coset Lagrangian. Therefore in the remainder of this section we will sketch some aspects of coset Lagrangians (see also [7–9]).

The nilpotent subalgebra \mathfrak{n} is actually the subalgebra

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda. \quad (3.3)$$

³Of course only the identity component of G/K is parameterized in this way.

Let us introduce a basis $\{H_1, \dots, H_l\}$ for \mathfrak{a} , where $l = \dim \mathfrak{a}$ is the rank of the real form \mathfrak{g} and let us fix a basis for \mathfrak{n} by the elements E_λ^I , where for fixed $\lambda \in \Sigma^+$ the index I runs from 1 to m_λ . The coset G/K can then be described by scalars ϕ_i , called dilatons, and by scalars A_λ^I , called axions, through $\mathcal{V} = \mathcal{V}_1 \mathcal{V}_2$ which is an element of G and where

$$\mathcal{V}_1 = \exp \frac{1}{2} \sum_{i=1}^l \phi_i H_i, \quad \mathcal{V}_2 = \exp \sum_{\lambda \in \Sigma^+} \sum_{I=1}^{m_\lambda} A_\lambda^I E_\lambda^I. \quad (3.4)$$

As we will see later it is sometimes more convenient to parameterize \mathcal{V}_2 slightly differently. We arrange the dilatons ϕ_i in a vector $\vec{\phi}$ and similar for H_i . For the restricted roots we define $\vec{\lambda}$ as the vector with components $\lambda_i = \lambda(H_i)$. From \mathcal{V} we can compute the Lie algebra valued one-form⁴

$$d\mathcal{V}\mathcal{V}^{-1} = \frac{1}{2} d\vec{\phi} \cdot \vec{H} + \sum_{\lambda \in \Sigma^+} \sum_{I=1}^{m_\lambda} e^{\vec{\lambda} \cdot \vec{\phi}} F_\lambda^I E_\lambda^I. \quad (3.5)$$

With every real form \mathfrak{g} goes a Cartan involution θ (see appendix A but also [12, 13]) which is +id on the compact subalgebra \mathfrak{k} . This Cartan involution is used to define a generalized transpose $\#$ in the (identity component of the) real group G as follows: if $O \in G$ and $O = \exp x$ for some $x \in \mathfrak{g}$, then $O^\# = \exp -\theta(x)$. In fact if $U \in K \subset G$, then we have $U^\# = U^{-1}$, which clarifies the name generalized transpose, since for $\text{SO}(n) \ni O$ we have $O^{-1} = O^T$. A general scalar coset action is of the form

$$S_{G/K} = \frac{1}{8} \int d^D x e \text{Tr} (\partial \mathcal{M} \partial \mathcal{M}^{-1}). \quad (3.6)$$

where the trace is in some representation and $\mathcal{M} = \mathcal{V}^\# \mathcal{V}$. Though \mathcal{M} is in a representation of the group, the trace in the action is actually in a Lie algebra representation and using that θ is an automorphism one can show that the action can be written as:

$$\begin{aligned} S_{G/K} &= -\frac{1}{4} \int d^D x e \left(\text{Tr} (\partial \mathcal{V} \mathcal{V}^{-1} \partial \mathcal{V} \mathcal{V}^{-1}) + \text{Tr} (\partial \mathcal{V} \mathcal{V}^{-1} (\partial \mathcal{V} \mathcal{V}^{-1})^\#) \right) \\ &= -\frac{1}{2} \int d^D x e \text{Tr} (\partial \mathcal{V} \mathcal{V}^{-1} \mathbb{P} \partial \mathcal{V} \mathcal{V}^{-1}), \end{aligned} \quad (3.7)$$

where $\mathbb{P} : \mathfrak{g} \rightarrow \mathfrak{g}$ denotes the projection operator defined by

$$\mathbb{P} : x \mapsto \frac{1}{2} (\mathbb{1} - \theta)x. \quad (3.8)$$

Hence \mathbb{P} is the identity on the non-compact part and zero on the compact part; it projects out the compact part. This is quite general for scalar coset Lagrangians; one starts with a representative V of the group G parameterized by scalars and writes

$$dV V^{-1} = Q + P, \quad Q \in \mathfrak{k}, P \in \mathfrak{p}. \quad (3.9)$$

⁴In deriving these formula one uses the Baker-Campbell-Hausdorff formulae, which can be found in appendix C.

Under a global G transformation $V \mapsto VM$, $M \in G$ the forms Q and P are invariant. From the relations 3.1 one finds that under a transformation $V \mapsto OV$, $O \in K$ we have

$$Q \mapsto dOO^{-1} + OQO^{-1}, \quad P \mapsto OPO^{-1}, \quad (3.10)$$

and thus Q is like a gauge field and P transforms covariant. Hence we can form the Lagrangian

$$L_{G/K} = -\frac{1}{2} \text{Tr}(P_\mu P^\mu), \quad (3.11)$$

which is precisely the same is

$$-\frac{1}{2} \text{Tr}(\mathbb{P}dVV^{-1}\mathbb{P}dVV^{-1}) = -\frac{1}{2} \text{Tr}(dVV^{-1}\mathbb{P}dVV^{-1}), \quad (3.12)$$

where the latter equality follows from $\text{Tr}(xy) \sim \text{Tr}_{\text{ad}}(\text{ad } x \text{ ad } y) = 0, \forall x \in \mathfrak{k}, y \in \mathfrak{p}$.

Another approach for coset Lagrangian is to start with the Lagrangian

$$L'_{G/K} = -\frac{1}{2} \text{Tr}(D_\mu VV^{-1}D^\mu VV^{-1}), \quad (3.13)$$

where the covariant derivative D contains a gauge field A_μ taking values in \mathfrak{k} and appears algebraically in the action. The gauge field can be eliminated by its equation of motion $\text{Tr}(A_\mu D^\mu VV^{-1}) = 0$, which precisely means that $D_\mu VV^{-1} \in \mathfrak{p}$. Hence the gauge field cancels the compact part in $\partial_\mu VV^{-1}$ giving thus the same Lagrangian: $L_{G/K} = L'_{G/K}$. If the scalar sector in the Lagrangian obtained by dimensional reduction matches the action 3.7 for the appropriate G/K , then indeed the scalars from the reduction parameterize the coset G/K . In the following section we will pursue this programme for the dimensionally reduced Heterotic supergravity.

3.2 Identifying restricted roots in the Lagrangian

The restricted roots are easily read off from the Lagrangian 2.17a to be \vec{B}_{ij} , \vec{A}_{ij} and \vec{C}_i with multiplicities $m(\vec{B}_{ij}) = m(\vec{A}_{ij}) = 1$ and $m(\vec{C}_i) = N$, while $m(2\vec{C}_i) = 0$. The simple restricted roots can be identified as follows: $\lambda_{d-i} \leftrightarrow \vec{B}_{i,i+1}$ and $\lambda_d \leftrightarrow \vec{C}_1$ and hence the rank of the coset is d . The dilaton coupling vectors are $d+1$ dimensional so one direction in this vector space should be redundant. In fact in [10] it is shown that indeed for $d < 6$ one can split of one component of the dilaton. We will take another approach; since it is known that in four dimensions the global symmetry group can be enlarged to an $\text{SL}(2; \mathbb{R}) \times \text{SO}(6, 6+N)$ we will embed the symmetry group already in the larger group $\text{SL}(2; \mathbb{R}) \times \text{SO}(d, d+N)$ which has rank $d+1$. It is easy to see that the inner product in the restricted root space is proportional to the inner product of the dilaton coupling vectors and thus the restricted root Dynkin diagram is

$$\begin{array}{ccccccc} \lambda_1 & & \lambda_2 & & \dots & & \lambda_{d-1} & \lambda_d \\ \circ & \text{---} & \circ & \text{---} & \dots & \text{---} & \circ & \Rightarrow & \circ \end{array}$$

and taking into account the multiplicities one can read off that the coset should be of the type BI or DI . This implies $G = \text{SO}(d, d+N)$, since the rank equals l for both cases and $N = 2(r-l)$ if $N+2d$ is even (type DI) and $N = 2(r-l) + 1$ if $N+2d$ is odd (type BI).

The rest of this section will thus be devoted to prove that indeed the scalar part of the Lagrangian 2.17a is an $\text{SO}(d, d + N)/\text{SO}(d) \times \text{SO}(d + N)$ coset Lagrangian. Some aspects of the explicit representations come in handy at this point. With every restricted root we can identify as many generators as the multiplicity and also we assemble $d + 1$ non-compact⁵ Cartan generators H_a , $0 \leq a \leq d$ in a vector where the $\mathfrak{sl}(2; \mathbb{R})$ Cartan generator is embedded as a linear combination. We therefore make the following identification:

$$\vec{B}_{ij} \leftrightarrow E_{ij}, \quad i < j, \quad [\vec{H}E_{ij}] = \vec{B}_{ij}E_{ij}, \quad (3.14a)$$

$$\vec{A}_{ij} \leftrightarrow R_{ij} = -R_{ji}, \quad [\vec{H}R_{ij}] = \vec{A}_{ij}R_{ij}, \quad (3.14b)$$

$$\vec{C}_i \leftrightarrow Y_{iI}, \quad 1 \leq I \leq N, \quad [\vec{H}Y_{iI}] = \vec{C}_iY_{iI}. \quad (3.14c)$$

The summation rules suggest that we take

$$\begin{aligned} [E_{ij}E_{kl}] &\sim \delta_{jk}E_{il} - \delta_{il}E_{kj}, & [E_{ij}R_{kl}] &\sim \delta_{ik}R_{jl} - \delta_{il}R_{jk}, \\ [E_{ij}Y_{kK}] &\sim \delta_{ik}Y_{jK}, & [Y_{iI}Y_{jJ}] &\sim M_{IJ}R_{ij}, \\ [Y_{iI}R_{kl}] &= 0, & [R_{ij}R_{kl}] &= 0, \end{aligned} \quad (3.15)$$

where M_{IJ} is an unknown matrix. Working out the Jacobi equations fixes the proportionality constants but not M_{IJ} , since this is related to a choice of basis in the subspace spanned by the Y_{iI} . Using the vector representation with the basis as in appendix B and working out the commutation relations gives $M_{IJ} = \delta_{IJ}$. Hence we have:

$$\begin{aligned} [E_{ij}E_{kl}] &= \delta_{jk}E_{il} - \delta_{il}E_{kj}, & [E_{ij}R_{kl}] &= -\delta_{ik}R_{jl} + \delta_{il}R_{jk}, \\ [E_{ij}Y_{kK}] &= -\delta_{ik}Y_{jK}, & [Y_{iI}Y_{jJ}] &= \delta_{IJ}R_{ij}, \\ [Y_{iI}R_{kl}] &= 0, & [R_{ij}R_{kl}] &= 0. \end{aligned} \quad (3.16)$$

The embedding in $\text{SL}(2; \mathbb{R}) \times \text{SO}(d, d + N)$ is done by embedding the vector representation $(2d + N) \times (2d + N)$ matrices X (see appendix B) of the positive restricted root vectors of $\mathfrak{so}(d, d + N)$ into $(2 + 2d + N) \times (2 + 2d + N)$ matrices as follows

$$X \mapsto \begin{pmatrix} 0_{2 \times 2} & 0_{2 \times (2d+N)} \\ 0_{(2d+N) \times 2} & X \end{pmatrix}, \quad (3.17)$$

and the Cartan generators are diagonal in this basis where the upper left block is proportional to the third Pauli matrix. We introduce for convenience the vector $\vec{C}_0 = \frac{1}{2}\sqrt{D-2}\vec{G}$ to get

$$\vec{C}_a \cdot \vec{C}_b = 2\delta_{ab}, \quad 0 \leq a, b \leq d; \quad \sum_{a=0}^d (\vec{C}_a)_m (\vec{C}_a)_n = 2\delta_{mn}. \quad (3.18)$$

Inspection of the commutation relations 3.16 and using the explicit vector representation we see that the Cartan generators H_a are diagonal with diagonal values

$$\begin{aligned} H_a &= \text{diag}((H_a)_0, -(H_a)_0, (H_a)_1, \dots, (H_a)_d, -(H_a)_1, \dots, -(H_a)_d, \underbrace{0, \dots, 0}_{N \text{ times}}), \\ (H_m)_a &= (-\vec{C}_a)_m, \quad 1 \leq a, m \leq d. \end{aligned} \quad (3.19)$$

⁵As much as is possible we will omit the subjective non-compact; a Cartan generator in this context is an element of \mathfrak{a} .

This implies that in the vector representation the traces are

$$\begin{aligned} \text{Tr}(H_a H_b) &= 2 \sum_{c=0}^d (\vec{C}_c)_a (\vec{C}_c)_b = 4\delta_{ab}, \\ \text{Tr}(E_{ij} E_{kl}^T) &= 2\delta_{ik} \delta_{jl}, \\ \text{Tr}(R_{ij} R_{mn}^T) &= 2\delta_{im} \delta_{jn}, \end{aligned} \tag{3.20}$$

and the others are zero. The traces in two different representations of a simple Lie algebra are proportional and thus for a semi-simple Lie algebra of two simple factors, there are two proportionality constants since both factors can be weighted differently. In our case we have fixed \vec{C}_0 and therefore we have fixed one more constant and this choice gives the right coset Lagrangian. The coset Lagrangian can be constructed using

$$\mathcal{V}_1 = \exp\left(\frac{1}{2}\vec{\Phi} \cdot \vec{H}\right), \tag{3.21a}$$

$$\mathcal{V}_2 = \cdots U_{24} U_{23} \cdots U_{14} U_{13} U_{12}, \quad U_{ij} = \exp(A_{ij} E_{ij}) \text{ no sum}, \tag{3.21b}$$

$$\mathcal{V}_3 = \exp\left(\sum_{i<j} B_{ij} R_{ij}\right) \tag{3.21c}$$

$$\Omega = \exp(\omega), \quad \omega = \sum_{iI} A_{iI} Y_{iI}, \tag{3.21d}$$

$$\mathcal{V} = \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \Omega, \tag{3.21e}$$

where A_{ij} , B_{ij} and A_{iI} are the axions and $\vec{\Phi}$ are the dilatons. Using some tricks as explained in appendix C one finds:

$$d\mathcal{V}_1 \mathcal{V}_1^{-1} = \frac{1}{2} d\vec{\Phi} \cdot \vec{H}, \tag{3.22a}$$

$$\mathcal{V}_1 d\mathcal{V}_2 \mathcal{V}_2^{-1} \mathcal{V}_1^{-1} = \sum_{i<j} F_{ij} e^{\frac{1}{2}\vec{\Phi} \cdot \vec{B}_{ij}} E_{ij}, \tag{3.22b}$$

$$\mathcal{V}_1 \mathcal{V}_2 d\mathcal{V}_3 \mathcal{V}_3^{-1} \mathcal{V}_2^{-1} \mathcal{V}_1^{-1} = \sum_{i<j} e^{\frac{1}{2}\vec{A}_{ij} \cdot \vec{\Phi}} dB_{mn} \Gamma_{mi} \Gamma_{nj}, \tag{3.22c}$$

$$\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 d\Omega \Omega^{-1} (\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3)^{-1} = \sum_{iI} e^{\frac{1}{2}\vec{C}_{iI} \cdot \vec{\Phi}} F_i^I Y_{iI} + \frac{1}{2} \sum_{i,j} e^{\frac{1}{2}\vec{A}_{ij} \cdot \vec{\Phi}} A_m^I \Gamma_{mi} F_j^I R_{ij}. \tag{3.22d}$$

Using that the action of θ in the vector representation becomes $\theta(x) = -x^T$ and using the properties of the traces one then indeed finds that the coset construction based on the commutation rules of the restricted root generators of $\mathfrak{so}(d, d + N)$ indeed gives the coset scalar Lagrangian as it appears in the action 2.17a. Though we embedded the global symmetry group into $\text{SL}(2; \mathbb{R}) \times \text{SO}(d, d + N)$, $\text{SL}(2; \mathbb{R})$ is not a part of the symmetry group if $D > 4$. This can be seen from the fact that there is no restricted root for $\text{SL}(2; \mathbb{R})$. This will change when the two-form can be dualized to a scalar; this can be done in four dimensions.

4. Analysis of dimensionally reduced symmetries

The fields appearing in the Lagrangian 2.17a do not all have the ‘right’ transformation properties, e.g. the Kaluza-Klein U(1)-gauge transformation also acts on other fields than the Kaluza-Klein vectors. The fields are not ‘diagonalized’ with respect to the symmetry transformations, but a field redefinition can achieve this. An alternative but equivalent approach is to slightly modify the reduction Ansatz in such a way that the difference between world and tangent indices is respected. Let us split up the world indices according to $\hat{\mu} \rightarrow (\mu, \alpha)$ and the tangent space indices according to $\hat{a} \rightarrow (a, i)$ and if more than one index of a kind is needed, the order of the alphabet will be used. The modification of the reduction Ansatz for the metric amounts to

$$h^i \rightarrow O^i_{\alpha}(dz^{\alpha} + V_{\mu}^{\alpha}dx^{\mu}), \quad O^i_{\alpha} = \delta^i_{\alpha} + A^i_{\alpha}, \quad (4.1)$$

where A^i_{α} are the axions with the distinction between flat and curved index expressed by using a different kind of letter. The metric Ansatz can now be written in terms of vielbeins by

$$\begin{aligned} \hat{e}_{\mu}^a &= e^{\frac{1}{4}\tilde{g}\cdot\tilde{\varphi}} e_{\mu}^a, & \hat{e}_{\mu}^i &= e^{\tilde{\gamma}_i\cdot\tilde{\varphi}} O^i_{\alpha} V_{\mu}^{\alpha}, \\ \hat{e}_{\alpha}^i &= e^{\tilde{\gamma}_i\cdot\tilde{\varphi}} O^i_{\alpha}, & \hat{e}_{\mu}^i &= 0. \end{aligned} \quad (4.2)$$

The Ansatz of the fields will now be made in a tangent space basis (see e.g. [11, 14]). So for the Kalb-Ramond field we write:

$$\hat{B} = \frac{1}{2}B_{ab}e^ae^b + B_{a\alpha}e^af^{\alpha} + \frac{1}{2}B_{\alpha\beta}f^{\alpha}f^{\beta}, \quad f^{\alpha} \equiv dz^{\alpha} + V_{\mu}^{\alpha}dx^{\mu}, \quad (4.3)$$

and for all other fields similar. Note that we still take A^i_{α} to be upper triangular. The most general diffeomorphism in ten dimensions compatible with the field Ansätze is

$$\delta x^{\mu} = -\xi^{\mu}(x), \quad \delta z^{\alpha} = \Lambda^{\alpha}_{\beta}z^{\beta} + \xi^{\alpha}(x), \quad (4.4)$$

where Λ^{α}_{β} is a constant matrix, hence a member of $\mathfrak{gl}(d; \mathbb{R}) \sim \mathbb{R} + \mathfrak{sl}(d; \mathbb{R})$ and from the discussion in [5] we know that \mathbb{R} -factor will combine with the higher dimensional scaling symmetry to an internal abelian symmetry. We will not be concerned with this symmetry. The ξ^{μ} -transformations give rise to lower dimensional general coordinate transformation, while the ξ^{α} -transformations act only on the Kaluza-Klein vectors V_{μ}^{α} , which transform under this transformation as U(1)-connections. Any field of the form $C_{\mu_1\dots\mu_p\alpha_1\dots\alpha_q}$ transforms as a q -tensor under the $\mathfrak{sl}(d; \mathbb{R})$ -transformations:

$$\delta C_{\mu_1\dots\mu_p\alpha_1\dots\alpha_q} = C_{\mu_1\dots\mu_p\beta\alpha_2\dots\alpha_q}\Lambda^{\beta}_{\alpha_1} + \text{other terms with } \alpha_1 \leftrightarrow \alpha_i. \quad (4.5)$$

The Kaluza-Klein vector transforms under $\mathfrak{sl}(d; \mathbb{R})$ as

$$\delta_{\Lambda} V_{\mu}^{\alpha} = -\Lambda^{\alpha}_{\beta} V_{\mu}^{\beta}. \quad (4.6)$$

The Kalb-Ramond transformations act on the two-form field as $\delta \hat{B} = d\hat{\Lambda}^{(1)}$, and the only transformation consistent with the reduction Ansatz acting on the scalar component of the two-form is

$$\delta_m \hat{B}_{\alpha\beta} = \delta_m B_{\alpha\beta} = m_{\alpha\beta}, \quad (4.7)$$

where $m_{\alpha\beta}$ is a constant anti-symmetric matrix. Consistent with the reduction Ansatz the scalars of the Yang-Mills vectors transform under a ten dimensional Yang-Mills transformation as

$$\delta_q \hat{A}_\alpha^I = \delta_q A_\alpha^I = q_\alpha^I, \tag{4.8}$$

where the parameters q_α^I are constants. To make the field strength $\hat{H} = d\hat{B} - \frac{1}{2}\hat{A}^I \hat{F}^I$ invariant the Kalb-Ramond field must also transform and the scalar part of the Kalb-Ramond field must transform as

$$\delta_q B_{\alpha\beta} = -q_{[\alpha}^I A_{\beta]}^I. \tag{4.9}$$

Calculating commutators of these transformations acting on the scalars we get

$$\begin{aligned} [\delta_{m_1}, \delta_{m_2}] &= 0, & [\delta_q, \delta_m] &= 0, \\ [\delta_{\Lambda_1}, \delta_{\Lambda_2}] &= \delta_{\Lambda_3}, & [\delta_\Lambda, \delta_q] &= -\delta_{q'}, \\ [\delta_{q_1}, \delta_{q_2}] &= \delta_{m_3}, & [\delta_\Lambda, \delta_m] &= \delta_{m'}, \\ \Lambda_3^{\alpha\beta} &= [\Lambda_1, \Lambda_2]^{\alpha\beta}, & q_\alpha^I &= q_\gamma^I \Lambda^\gamma_\alpha, \\ m_3 &= q_1^I q_2^I - q_2^I q_1^I, & m'_{\alpha\beta} &= m_{\alpha\gamma} \Lambda^\gamma_\beta + m_{\gamma\beta} \Lambda^\gamma_\alpha. \end{aligned} \tag{4.10}$$

We identify generators with these transformations according to $\Lambda^\alpha_\beta \leftrightarrow E_{\alpha\beta}$, $m_{\alpha\beta} \leftrightarrow R_{\alpha\beta} = -R_{\beta\alpha}$ and $q_\alpha^I \leftrightarrow Y_\alpha^I$. Note that to keep A^I_α upper triangular, Λ must be an upper triangular matrix and hence $E_{\alpha\beta}$ only is nonzero for $\alpha < \beta$. From the given commutators of transformations we calculate the commutation relations of the generators and find

$$\begin{aligned} [E_{\alpha\beta}, E_{\gamma\delta}] &= \delta_{\beta\gamma} E_{\alpha\delta} - \delta_{\alpha\delta} E_{\gamma\beta}, & [R_{\alpha\beta}, R_{\gamma\delta}] &= 0, \\ [R_{\alpha\beta}, Y_\gamma^I] &= 0, & [Y_\alpha^I, Y_\beta^J] &= 2\delta^{IJ} R_{\alpha\beta}, \\ [E_{\alpha\beta}, Y_\gamma^I] &= -\delta_{\alpha\gamma} Y_\beta^I, & [E_{\alpha\beta}, R_{\gamma\delta}] &= -\delta_{\alpha\gamma} R_{\beta\delta} + \delta_{\alpha\delta} R_{\beta\gamma}, \end{aligned} \tag{4.11}$$

which is up to a scaling of $R_{\alpha\beta}$ the same as the algebra given in section 3.2. Hence the dimensionally reduced scalar symmetry transformations are generated by the positive restricted root subalgebra \mathfrak{n} of $\mathfrak{so}(d, d + N)$.

5. Maximal scalar manifolds and dualizations

The concept of a maximal scalar manifold was also discussed in [5]. In D dimensions a $(D - 2)$ -form can be dualized to a scalar; when this is done, the so-called maximal scalar manifold is obtained. For the Heterotic supergravities this can be done for $D = 4$ and $D = 3$. In many cases the global symmetry group is enlarged. If a form is dualized to a scalar the Lagrangian obtains a topological term, which we will ignore, and the dilaton coupling vector appears with a different sign. In four dimensions the dualization of the two-form gives a term

$$-\frac{1}{2} e^{-\vec{A} \cdot \vec{\Phi}} (\partial\chi)^2 \tag{5.1}$$

in the Lagrangian and so we can relate to the extra axion χ a positive root generator \mathcal{S} . Since the vector \vec{A} has no non-trivial summation rules, \mathcal{S} will commute with all of them except for the Cartan generators. Hence only the commutator

$$[\vec{H}\mathcal{S}] = -\vec{A}\mathcal{S} = \vec{G}\mathcal{S} \tag{5.2}$$

is non-zero. One can compare the diagonalization procedure of [10] to splitting of this commutator of the $\mathfrak{so}(d, d + N)$ algebra and hence the diagonalization is also a diagonalization in the Cartan subalgebra of $\mathfrak{sl}(2; \mathbb{R}) \oplus \mathfrak{so}(d, d + N)$. The restricted root diagram now consists of two parts, one part is the same as in section 3.2 with $d = 6$, the other is that of $SL(2; \mathbb{R})/SO(2) \cong SO(2, 1)_0/SO(2)$ ⁶. A coset construction reveals that indeed the global symmetry group is $SL(2; \mathbb{R}) \times SO(6, 6 + N)$ (this is also standard, see e.g. [9, 10]). The dualized action can be further reduced to $D = 3$ and in $D = 3$ all vector fields can be dualized. One obtains the following dilaton coupling vectors

$$\begin{aligned} \vec{B}_{ij}, \quad -\vec{B}_i = \vec{F}_i, \quad \vec{A}_{ij}, \quad \vec{C}_i, \quad \frac{1}{2}\vec{G} \quad 1 \leq i, j \leq 7; \\ -\vec{A}_i, \quad 1 \leq i \leq 6, \quad -\vec{A}_\chi \equiv -\vec{A}_7, \end{aligned} \tag{5.3}$$

where \vec{A}_χ results from the vector \vec{A} in the dilaton coupling of χ in four dimensions and we have

$$\begin{aligned} \vec{B}_{ij} = -\vec{F}_i + \vec{F}_j, \quad \vec{A}_{ij} = \vec{F}_i + \vec{F}_j - \vec{G}, \\ \vec{C}_i = \vec{F}_i - \frac{1}{2}\vec{G}, \quad -\vec{A}_i = \vec{F}_i - \vec{G}. \end{aligned} \tag{5.4}$$

giving rise to the set of simple restricted roots $\{\vec{B}_{i,i+1}, \vec{C}_1, -\vec{A}_7\}$. We see that the only difference with the naively expected $SO(7, 7 + N)$ symmetry group, there is an additional restricted root $-\vec{A}_7$ which only has a nonzero inner product with \vec{B}_{67} . So we could call it \vec{B}_{78} and then we see that the coset structure is the same as when we would have reduced over 8 dimensions instead of just 7, and hence the global symmetry group is $SO(8, 8 + N)$, which is a known result.

6. Conclusions

In this paper we outlined a general method for recognizing the global symmetry groups which appear after dimensional reduction in extended lower dimensional supergravities. The method can be broken down into three steps. First one does a circle by circle dimensional reduction as described in section 2. As a second step one identifies the dilatonic coupling vectors as restricted roots and then draws the associated Dynkin diagram and counts the multiplicities; this fixes the coset. As a third step one constructs a scalar coset Lagrangian for the coset found in step two and compares the result to that of the dimensional reduction.

In this paper we showed that the first and second step uniquely fix the coset. The complete classification of real forms of simple Lie algebras turns out to be indispensable and powerful. The third step is to verify that the scalar part of the Lagrangian obtained from the reduction coincides with a scalar coset Lagrangian of the coset found in the second step. The third step can be involved but contains no difficulties of principles. The method has been applied to the dimensional reduction of Heterotic supergravity, where the Lie algebra of G is a non-split real form.

⁶The index 0 means the component connected to the identity.

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A. Real forms of Lie algebras: a quick reference

Every semi-simple real Lie algebra \mathfrak{g} admits a Cartan involution θ which is an involutive automorphism, such that the bilinear form $B_\theta(x, y) = -B(x, \theta y)$ is positive definite and B denotes the Cartan-Killing form. The $+1(-1)$ eigenspace is denoted $\mathfrak{k}(\mathfrak{p})$ and we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and schematically the commutation relations are as in eq. (3.1). With respect to the inner product B_θ we have $(\text{ad } x)^\dagger = -\text{ad } \theta x$ and thus a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ induces a decomposition of \mathfrak{g} into eigenspaces w.r.t. \mathfrak{a} . This is a restricted root decomposition analogous to the usual root decomposition and we have the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda, \quad \mathfrak{g}_0 = \mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a}), \tag{A.1}$$

$$[Hx_\lambda] = \lambda(H)x_\lambda, \quad \forall H \in \mathfrak{a}, x_\lambda \in \mathfrak{g}_\lambda,$$

where $\Sigma \subset \mathfrak{a}^*$ denotes the set of restricted roots and $Z_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of \mathfrak{a} in \mathfrak{k} . The Iwasawa decomposition of the Lie algebra \mathfrak{g} is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad \mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda. \tag{A.2}$$

Let \mathfrak{t} be a maximal torus in $Z_{\mathfrak{k}}(\mathfrak{a})$. Complexifying the Lie algebra to $\mathfrak{g}^{\mathbb{C}}$ a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ can be found by extending $\mathfrak{a}^{\mathbb{C}}$ with the complexified maximal torus $\mathfrak{t}^{\mathbb{C}}$. The set of roots is denoted Δ and the roots are real on $\mathfrak{a} \oplus i\mathfrak{t}$. The action of θ is extended by linearity and we see that $-\theta$ acts as complex conjugation on the real Cartan subalgebra $\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{t}$. The action of θ on a root α is defined by $(\theta\alpha)(h) = \alpha(\theta h)$, $h \in \mathfrak{h}$. A root α is called real if $\alpha|_{i\mathfrak{t}} \equiv 0$, and imaginary if $\alpha|_{\mathfrak{a}} = 0$ and complex if it is neither real nor imaginary. One can find an ordering of the roots in which \mathfrak{a} is taken before $i\mathfrak{t}$ implying that θ permutes the simple roots. A Satake diagram is a diagram in which the imaginary simple roots are colored, while the real and complex are not but the two-element orbits of θ is denoted by arrows. The projection of a root α to its restriction to \mathfrak{a} , denoted $\bar{\alpha}$, is easily seen to be $\bar{\alpha} = \frac{1}{2}(\alpha - \theta\alpha)$. It is easy to see that $\bar{\alpha}(i\mathfrak{t}) = 0$ and $\bar{\alpha} \in \Sigma$. If λ is a restricted root, then we define $\Delta_\lambda = \{\alpha \in \Delta | \bar{\alpha} = \lambda\}$. The number of roots in this subspace is called the multiplicity of the restricted root λ : $m_\lambda = \text{Card}\Delta_\lambda$. Since for restricted roots the root system can be non-reduced (i.e. if $\lambda \in \Sigma$, also 2λ can be in Σ) also $m_{2\lambda}$ can be non-zero.

A simple restricted root is a positive restricted root which can not be written as the sum of two positive restricted roots. The set of simple restricted roots $\{\lambda_1, \dots, \lambda_l\}$ contains

$l = \dim \mathfrak{a}$ elements and gives rise to a restricted root Dynkin diagram. It is a theorem that if the restricted root system (i.e. the restricted root Dynkin diagram) and the multiplicities m_{λ_i} and $m_{2\lambda_i}$ of the simple restricted roots λ_i are known, the real form of the simple Lie algebra is uniquely determined. This enables us to list all of them and since the compact form always exists, we only list all the non-compact real forms in appendix D. In these tables the simple roots α_i , $1 \leq i \leq r = \dim_{\mathbb{R}} \mathfrak{h}$ are related to the simple restricted roots λ_i by $\lambda_i = \bar{\alpha}_i$. The table contains information that can be found in [12]. The cosets G/K with K maximally compact are also put in the table. The Satake diagrams are there to show which simple roots of the original Dynkin diagram survive the projection to the simple restricted roots.

B. Some details of the Lie algebra $\mathfrak{so}(d, d + N)$.

In this section we use the same notation as in appendix A. With the use of the η -matrix

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_{d \times d} & 0 \\ \mathbb{1}_{d \times d} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{N \times N} \end{pmatrix} \quad (\text{B.1})$$

it is easy to work out the constraint $X^T \eta + \eta X = 0$ for a $(2d + N) \times (2d + N)$ -matrix X . The Cartan involution θ will then be defined by its action in this vector representation $\mathfrak{so}(d, d + N)_V$ by

$$\theta(X) \mapsto -X^\dagger = -X^T; X \in \mathfrak{so}(d, d + N)_V, \quad (\text{B.2})$$

and since the vector representation is faithful, this is indeed an automorphism of the Lie algebra. The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the eigenspaces of θ is easily found. A maximal abelian subspace \mathfrak{a} of \mathfrak{p} where θ is $-\text{id}$ can be found to be spanned by the diagonal matrices of the form

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{B.3})$$

where a is $d \times d$ diagonal matrix. We can set up an isomorphism between \mathbb{R}^d and \mathfrak{a} by the isomorphism (which is a vector-space isomorphism) $\phi : \mathbb{R}^d \rightarrow \mathfrak{a}$ defined by

$$\phi : \vec{a} = (a_1, \dots, a_d) \mapsto \text{diag}(a_1, \dots, a_d, -a_1, \dots, -a_d, \underbrace{0, \dots, 0}_{N \text{ zeros}}), \quad (\text{B.4})$$

Now define the following basis in the dual of \mathbb{R}^d

$$\tilde{\lambda}_i : \vec{a} = (a_1, \dots, a_d) \mapsto a_i \in \mathbb{R}. \quad (\text{B.5})$$

The restricted roots of $\mathfrak{so}(d, d + N)$ with respect to \mathfrak{a} can be calculated to be

$$0, \pm \lambda_i \equiv \pm \phi \circ \tilde{\lambda}_i \circ \phi^{-1}, \pm(\lambda_i - \lambda_j) \ i \neq j, \pm(\lambda_i + \lambda_j) \ i \neq j, \quad (\text{B.6})$$

and denoting the restricted root spaces by \mathfrak{g}_λ for a restricted root λ , we have $\dim \mathfrak{g}_{\pm\lambda_i} = N$, $\dim \mathfrak{g}_{\pm(\lambda_i \pm \lambda_j)} = 1$. We can see that $E_{ij} \in \mathfrak{g}_{\lambda_i - \lambda_j}$, $R_{ij} \in \mathfrak{g}_{-\lambda_i - \lambda_j}$ and $Y_{iI} \in \mathfrak{g}_{-\lambda_i}$ where

$$E_{ij} = \begin{pmatrix} e_{ij} & 0 & 0 \\ 0 & -e_{ji} & 0 \\ 0 & 0 & 0 \end{pmatrix}, R_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ \beta_{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y_{iI} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma_{iI} \\ -\gamma_{iI}^T & 0 & 0 \end{pmatrix}, \quad (\text{B.7})$$

with

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl}, \quad 1 \leq i, j, k, l \leq d, \quad (\text{B.8a})$$

$$\beta_{ij} = e_{ij} - e_{ji}, \quad 1 \leq i < j \leq d, \quad (\text{B.8b})$$

$$E_{ij} = e_{ij}, \quad 1 \leq i < j \leq d, \quad (\text{B.8c})$$

$$(\gamma_{iI})_{kK} = \delta_{ik}\delta_{IK}, \quad 1 \leq i, k \leq d, 1 \leq I, K \leq N. \quad (\text{B.8d})$$

Having chosen a sense of positivity and calling Σ^+ the set of positive roots, the Iwasawa decomposition is then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad \mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda. \quad (\text{B.9})$$

We see that we can chose \mathfrak{n} to be the span of the union of the sets $\{E_{ij} : 1 \leq i < j \leq d\}$, $\{R_{ij} = -R_{ji} : 1 \leq i, j \leq d\}$ and $\{Y_{iI} : 1 \leq i \leq d, 1 \leq I \leq N\}$. Though some minus signs may favor the name negative root part, this is just merely a matter of choice. A quick calculation reveals:

$$\begin{aligned} [E_{ij}Y_{kK}] &= -\delta_{ik}Y_{jK}, & [Y_{iI}Y_{jJ}] &= \delta_{IJ}R_{ij}, \\ [E_{ij}E_{kl}] &= \delta_{jk}E_{il} - \delta_{il}E_{kj}, & [E_{ij}R_{kl}] &= -\delta_{ik}R_{jl} + \delta_{il}R_{jk}, \\ [E_{ij}Y_{kK}] &= -\delta_{ik}Y_{jK}, & [Y_{iI}Y_{jJ}] &= \delta_{IJ}R_{ij}, \\ [Y_{iI}R_{kl}] &= 0, & [R_{ij}R_{kl}] &= 0. \end{aligned} \quad (\text{B.10})$$

C. Baker-Campbell-Hausdorff and tricks

Useful formulae are

$$e^X Y e^{-X} = \text{Ad}(e^X)Y = e^{\text{ad}X}Y = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \dots, \quad (\text{C.1a})$$

$$(\text{de}^X) e^{-X} = dX + \frac{1}{2}[X, dX] + \frac{1}{6}[X, [X, dX]] + \dots, \quad (\text{C.1b})$$

for explanations see [15]. In deriving the formulas 3.22b-3.22d it is handy if one uses that in the fundamental representation of the E_{ij} we have $(\mathcal{V}_2)_{ij}^{-1} = \Gamma_{ij}$, so we get $d(\mathcal{V}_2)_{jk}\Gamma_{km} = dA_{jk}\Gamma_{km} = F_{jm}$. The fundamental representation is faithful and hence it holds for all representations, hence for the Lie algebra. It is handy to note that E_{ij} acts on the Y_{iI} as a linear transformation in the vector space spanned by the Y_i - omitting the I -index since it is in this case a spectator:

$$\text{ad } E_{ij}(Y_n) = -\delta_{in}Y_j = \sum_m (\text{ad } E_{ij})_{nm} Y_m \Rightarrow (\text{ad } E_{ij})_{mn} = -\delta_{im}\delta_{jn}. \quad (\text{C.2})$$

The representing matrix of $\text{ad } E_{ij}$ squares to zero giving $\text{ad } E_{ij} \circ \text{ad } E_{ij}(Y_n) = 0$. One can easily proceed via

$$e^{A_{ij}E_{ij}}Y_n e^{-A_{ij}E_{ij}} = \text{Ad}(e^{A_{ij}E_{ij}})(Y_n) = e^{\text{ad } A_{ij}E_{ij}}(Y_n) = (\mathbb{1} + \text{ad } A_{ij}E_{ij})_{nm} Y_m. \quad (\text{C.3})$$

The generators R_{mn} are in the tensor-product representation of this representation under the action of $\text{ad } E_{ij}$ and hence one finds:

$$\mathcal{V}_2 R_{mn} \mathcal{V}_2^{-1} = \Gamma_{mp} \Gamma_{nq} R_{pq}. \quad (\text{C.4})$$

D. Tables

Satake Diagram	Restricted Root Diagram	Type
		AI: $\text{SL}(n; \mathbb{R})/\text{SO}(n)$ $l = r = n - 1$
		AII: $\text{SU}^*(2n)/\text{Sp}(n)$ $l = 2r - l = n - 1$
		AIII: $\text{SU}(p, q)/\text{S}(U_p \times U_q)$ $l = \min(p, q), r = p + q - 1$ If $2 \leq l \leq r/2$
		AIV: $\text{SU}(n, 1)/\text{SU}(n)$ $l = r + 1 = n$
		BI: $\frac{\text{SO}(p, q)}{\text{SO}(p) \times \text{SO}(q)}$; $p + q$ odd $2 \leq l = \min(p, q) \leq r$
		BII: $\text{SO}(2p, 1)/\text{SO}(2p)$ $l = 1, r = p$

Table 1: continued on next page.

Satake Diagram	Restricted Root Diagram	Type
$\alpha_1 \text{---} \alpha_2 \text{---} \dots \text{---} \alpha_{r-1} \alpha_r$ 	$\lambda_1 \text{---} \lambda_2 \text{---} \dots \text{---} \lambda_{r-1} \lambda_r$ 	CI: $\text{Sp}(n, \mathbb{R})/\text{U}(n)$ $l = r = n$
$\alpha_1 \text{---} \alpha_2 \text{---} \bullet \text{---} \dots \text{---} \alpha_{2l} \text{---} \bullet \text{---} \dots \text{---} \bullet \leftarrow \bullet$ 	$\lambda_2 \text{---} \lambda_4 \text{---} \dots \text{---} \circ \Rightarrow \lambda_{2l}$ 	CII: $\frac{\text{Sp}(p, q)}{\text{S}(p) \times \text{Sp}(q)}$ $l = \min(p, q)$ If $1 \leq l \leq \frac{1}{2}(r - l)$
$\alpha_1 \text{---} \alpha_2 \text{---} \bullet \text{---} \dots \text{---} \circ \text{---} \bullet \leftarrow \alpha_{2l}$ 	$\lambda_2 \text{---} \lambda_4 \text{---} \dots \text{---} \circ \leftarrow \lambda_{2l}$ 	If $2 \leq l = r/2$
$\alpha_1 \text{---} \dots \text{---} \alpha_l \text{---} \bullet \text{---} \dots \text{---} \bullet \begin{matrix} \nearrow \bullet \\ \searrow \bullet \end{matrix}$ 	$\lambda_1 \text{---} \lambda_2 \text{---} \dots \text{---} \lambda_{r-1} \lambda_r$ 	DI: $\frac{\text{SO}(p, q)}{\text{SO}(p) \times \text{SO}(q)}$, $p + q$ even $l = \min(p, q)$; If $2 \leq l \leq r - 2$
$\alpha_1 \text{---} \dots \text{---} \alpha_{l-1} \begin{matrix} \nearrow \alpha_l \\ \searrow \alpha_{l+1} \end{matrix}$ 	$\lambda_1 \text{---} \lambda_2 \text{---} \dots \text{---} \lambda_{r-1} \lambda_r$ 	If $r = l + 1$.
$\alpha_1 \text{---} \dots \text{---} \alpha_{l-2} \begin{matrix} \nearrow \alpha_{l-1} \\ \searrow \alpha_l \end{matrix}$ 	$\lambda_1 \text{---} \dots \text{---} \lambda_{l-2} \begin{matrix} \nearrow \lambda_{l-1} \\ \searrow \lambda_l \end{matrix}$ 	If $r = l$.
$\alpha_1 \text{---} \bullet \text{---} \dots \text{---} \bullet \begin{matrix} \nearrow \bullet \\ \searrow \bullet \end{matrix}$ 	λ_1 	DII: $\frac{\text{SO}(2r-1, 1)}{\text{SO}(2r-1)}$ $l = 1$
$\alpha_1 \text{---} \alpha_2 \text{---} \bullet \text{---} \dots \text{---} \alpha_{r-2} \begin{matrix} \nearrow \alpha_{r-1} \\ \searrow \alpha_r \end{matrix}$ 	$\lambda_2 \text{---} \lambda_4 \text{---} \dots \text{---} \circ \leftarrow \lambda_{2l}$ 	DIII: $\frac{\text{SO}^*(2n)}{\text{U}(n)}$ $l = [n/2]$ If $r = 2l$
$\alpha_1 \text{---} \alpha_2 \text{---} \bullet \text{---} \dots \text{---} \alpha_{r-2} \begin{matrix} \nearrow \alpha_{r-1} \\ \searrow \alpha_r \end{matrix}$ 	$\lambda_2 \text{---} \lambda_4 \text{---} \dots \text{---} \circ \Rightarrow \lambda_{2l}$ 	If $r = 2l + 1$

Table 1: continued on next page.

Satake Diagram	Restricted Root Diagram	Type
		EI : $E_{6(6)}/\text{Sp}(4)$. $l = r = 6$
		EII : $\frac{E_{6(2)}}{\text{SU}(6) \times \text{SU}(2)}$. $l = 4$
		EIII : $\frac{E_{6(-14)}}{\text{SO}(10) \times \text{U}(1)}$. $l = 2$
		EIV : $\frac{E_{6(-24)}}{F_4}$. $l = 2$
		EV : $E_{7(7)}/\text{SU}(8)$. $l = r = 7$
		EVI : $\frac{E_{7(-5)}}{\text{SO}(12) \times \text{SU}(2)}$. $l = 4$
		EVII : $\frac{E_{7(-25)}}{E_6 \times \text{U}(1)}$. $l = 3$
		EVIII : $E_{8(8)}/\text{SO}(16)$. $l = r = 8$
		EIX : $\frac{E_{8(-24)}}{E_7 \times \text{SU}(2)}$. $l = 4$
		FI : $\frac{F_{4(4)}}{\text{Sp}(3) \times \text{SU}(2)}$. $r = l = 4$
		FII : $F_{4(-20)}/\text{SO}(9)$. $l = 1$

Table 1: continued on next page.

Satake Diagram	Restricted Root Diagram	Type
$\circ \Rightarrow \circ$	$\circ \Rightarrow \circ$	$\mathbf{G}: \frac{G_{2(2)}}{\mathrm{SU}(2) \times \mathrm{SU}(2)}$ $l = r = 2$

Table 1: Satake diagrams and restricted root diagrams and associated cosets G/K .

Type		m_{λ_i}	$m_{2\lambda_i}$
AI	$\forall i$	1	0
AII	$\forall i$	4	0
AIII ; $2 \leq l \leq \frac{r}{2}$	$i < l$	2	0
	$i = l$	$2(r - 2l + 1)$	1
AIII ; $r = 2l - 1$	$i < l$	2	0
	$i = l$	1	0
AIV		$2(r - 1)$	1
BI	$i < l$	1	0
	$i = l$	$2(r - l) + 1$	0
BII	$\forall i$	$2r - 1$	0
CI	$\forall i$	1	0
CII ; $1 \leq l \leq \frac{1}{2}(r - 1)$	$i < 2l$	4	0
	$i = 2l$	$4(r - 2l)$	3
CII ; $2 \leq l = \frac{1}{2}r$	$i < 2l$	4	0
	$i = 2l$	3	0
DI ; $2 \leq l \leq r - 2$	$i < l$	1	0
	$i = l$	$2(r - l)$	0
DI ; $l = r - 1$	$i < l$	1	0
	$i = l$	2	0
DI ; $l = r$	$\forall i$	1	0
DII	$\forall i$	1	0
DIII ; $r = 2l$	$i < 2l$	4	0
	$i = 2l$	1	0
DIII ; $r = 2l + 1$	$i < 2l$	4	1
EI	$\forall i$	1	0
EII	$i = 2, 4$	1	0
	$i = 1, 3$	2	0
EIII	$i = 1$	8	1
	$i = 2$	6	0
EIV	$\forall i$	8	0
EV	$\forall i$	1	0
EVI	$i = 1, 3$	1	0
	$i = 2, 4$	4	0

Table 2: continued on next page.

Type		m_λ	$m_{2\lambda}$
EVII	$i = 1, 6$	8	0
	$i = 7$	1	0
EVIII	$\forall i$	1	0
EIX	$i = 1, 6$	8	0
	$i = 7, 8$	1	0
FI	$\forall i$	1	0
FII	$\forall i$	8	7
GI	$\forall i$	1	0

Table 2: Multiplicities of the restricted simple roots

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